MTHA6005Y Mathematics Project 2023/24

Representation theory of $\mathfrak{sl}_2(\mathbb{C})$

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Introduction

Background and Motivation

Lie groups are groups with special properties such that we can define and perform calculus-like operations, like differentiation, over the group. As their name suggests, Lie algebras are intimately related to Lie groups — every Lie group has an associated Lie algebra, though the converse is not always true. Working with a Lie group's associated Lie algebra is often easier than working with the group directly, and understanding these Lie algebras can enlighten us with information about its underlying Lie group.

Representation theory, on the other hand, looks to study objects, like Lie algebras, by defining functions that send elements of the object to linear maps over a vector space. These functions are called *representations* of the object. Hence, one of the obvious goals in representation theory is to classify the representations of a given object.

In this report, we study the classification of finite-dimensional representations of the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$. This is the smallest non-trivial member of a class of Lie algebras known as complex semisimple Lie algebras. We isolate this particular example because understanding the representation theory of $\mathfrak{sl}_2(\mathbb{C})$ can provide us with significant insights into classifying the representations of other complex semisimple Lie algebras.

Applications

Representation theory has myriad applications, so here we'll list just a few. In chemistry, representations of point groups aid in classifying molecules based on their symmetries [7, Chapter 4]. In computer science, quiver representations offer an algebraic depiction of neural networks [1]. Finally, in physics, quantum systems are often representations of the Lie group *Spin* (3). This group's corresponding Lie algebra, $\mathfrak{spin}(3)$, is isomorphic to the Lie algebra $\mathfrak{so}_3(\mathbb{R})$, and the complexification of $\mathfrak{so}_3(\mathbb{R})$ is isomorphic to $\mathfrak{sl}_2(\mathbb{C})$ [8, Chapter 6].

Report Overview

This report is split into three sections. In the first section, we'll introduce Lie algebras, in particular $\mathfrak{sl}_2(\mathbb{C})$ and $\mathfrak{gl}(V)$, and the homomorphism $\Psi_d : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V_d)$. In the following section, we'll introduce Lie algebra representations and Lie modules, and we'll prove that the $\mathfrak{sl}_2(\mathbb{C})$-module V_d is irreducible. Then, in the final section, we'll prove that every finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module is isomorphic to a V_d module and state a special case of Weyl's Theorem which classifies the completely reducible $\mathfrak{sl}_2(\mathbb{C})$-modules.

1 Lie algebras

1.1 Bilinear maps

To set the scene, a Lie algebra is a vector space V over a field F together with a particular map from $V \times V \to V$ satisfying certain conditions. One these conditions is that the map must be *bilinear*, in which case we call it a *bilinear map*.

Definition 1.1. (Adapted from [3, p.1]) Let F be a field and X, Y, and Z be F-vector spaces. Then, consider a map $* : X \times Y \to Z$ given by $(x, y) \mapsto x * y$. By fixing $x \in X$ and $y \in Y$, if for any $\alpha, \beta \in F$, any $x_1, x_2 \in X$, and any $y_1, y_2 \in Y$, we have that

$$(\alpha x_1 + \beta x_2) * y = \alpha (x_1 * y) + \beta (x_2 * y); \tag{BM1}$$

$$x * (\alpha y_1 + \beta y_2) = \alpha (x * y_1) + \beta (x * y_2), \tag{BM2}$$

then we call $*$ a *bilinear map* and refer to $*$ as *bilinear*.

We'll now provide an example of a bilinear map. Given a finite-dimensional F-vector space V, we let $\mathfrak{gl}(V)$ denote the F-vector space of linear maps from V to V. Defining $* : \mathfrak{gl}(V) \times \mathfrak{gl}(V) \to \mathfrak{gl}(V)$ by $f * g := (f \circ g) - (g \circ f)$, we have the following lemma.

Lemma 1.2. $*$ is a bilinear map.

Proof. Let $f, g \in \mathfrak{gl}(V)$ be fixed and let $\alpha, \beta \in F$ be arbitrary. Then, for any $f_1, f_2 \in \mathfrak{gl}(V)$, we have that

$$\begin{aligned}(\alpha f_1 + \beta f_2) * g &= ((\alpha f_1 + \beta f_2) \circ g) - (g \circ (\alpha f_1 + \beta f_2)) \\ &= ((\alpha f_1) \circ g) + ((\beta f_2) \circ g) - (g \circ (\alpha f_1)) - (g \circ (\beta f_2)) \\ &= \alpha (f_1 \circ g) + \beta (f_2 \circ g) - \alpha (g \circ f_1) - \beta (g \circ f_2) \\ &= \alpha ((f_1 \circ g) - (g \circ f_1)) + \beta ((f_2 \circ g) - (g \circ f_2)) \\ &= \alpha (f_1 * g) + \beta (f_2 * g).\end{aligned}$$

Thus, (BM1) is satisfied. Following a similar procedure, we have that

$$f * (\alpha g_1 + \beta g_2) = \alpha (f * g_1) + \beta (f * g_2)$$

for any $g_1, g_2 \in \mathfrak{gl}(V)$, so (BM2) is satisfied. Hence, $*$ is bilinear. □

Remark 1.3. $\mathfrak{gl}(V)$ will reveal itself to be a very important vector space later in this report. We will mention here that if $\dim(V) = n \geq 1$ and we fix the basis $\{e_1, \ldots, e_n\}$ for V, then $\{\theta_{i,j} : 1 \leq i, j \leq n\}$ forms a basis for $\mathfrak{gl}(V)$, where $\theta_{i,j} : V \to V$ is defined by

$$\theta_{i,j}(e_k) := \begin{cases} e_j, & \text{if } k = i; \\ 0_V, & \text{if } k \neq i; \end{cases}$$

and extending linearly. Hence, $\mathfrak{gl}(V)$ has dimension n^2, so it is isomorphic to $\mathfrak{gl}_n(F)$ — the vector space of $n \times n$ matrices over F. Moreover, we have an analogous bilinear map given by $XY - YX$, as stated in [2, p.3]. Due to the presence of this isomorphism, we can (and will) use $\mathfrak{gl}(V)$ and $\mathfrak{gl}_n(F)$ interchangeably.

1.2 Lie algebras

Having introduced bilinear maps, we can now define a *Lie algebra*.

Definition 1.4. [2, p.1] A *Lie algebra* over a field F is an F-vector space L together with a bilinear map $L \times L \to L$ given by $(a, b) \mapsto [a, b]$, known as the *Lie bracket*, which satisfies the following properties:

$$[a, a] = 0_L \text{ for any } a \in L; \tag{LA1}$$

$$[a, [b, c]] + [b, [c, a]] + [c, [a, b]] = 0_L \text{ for any } a, b, c \in L. \tag{LA2}$$

NB: We will sometimes write $[-, -]$ to refer to a (potentially arbitrary) Lie bracket on L. Furthermore, we'll often just say that L is a Lie algebra — omitting the mention of the field F and definition of the Lie bracket.

Before providing some examples of Lie algebras, we will first state a useful proposition pertaining to the Lie bracket.

Remark 1.5. [2, p.1] Let L be a Lie algebra over a field F and $x, y \in L$. Then, by (LA1), we have that $[x + y, x + y] = 0_L$. By applying (BM1) and (BM2), we obtain

$$[x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = 0_L.$$

Now, $[x, x] = [y, y] = 0_L$ by (LA1), hence we have that $[x, y] + [y, x] = 0_L$, which implies that $[x, y] = -[y, x]$.

We'll begin our examples with a class of Lie algebras known as *abelian* Lie algebras.

Example 1.6. [4, p.4] Given any F-vector space L, we can equip it with the Lie bracket $[x, y] = 0_L$ for all $x, y \in L$. By doing so, we have that (BM1), (BM2), (LA1), and (LA2) are trivially satisfied, so L is a Lie algebra — referred to as an *abelian* Lie algebra.

For our next example, which is less trivial, we'll show that $\mathfrak{gl}(V)$ together with the bilinear map $*$ from Lemma 1.2 forms a Lie algebra.

Proposition 1.7. [2, Exercise 1.3] $\mathfrak{gl}(V)$ with Lie bracket given by

$$[f, g] := (f \circ g) - (g \circ f)$$

is a Lie algebra.

Proof. We have that $[-,-]$ is bilinear by Lemma 1.2, so it remains to show that (LA1) and (LA2) are satisfied. Firstly, let $f \in \mathfrak{gl}(V)$. Then, $[f,f] = (f \circ f) - (f \circ f) = 0$. Hence, (LA1) is satisfied. Next, let $f, g, h \in \mathfrak{gl}(V)$. Then, by dropping the use of $\circ$ for brevity, we have that

$$\begin{aligned} &[f,[g,h]] + [g,[h,f]] + [h,[f,g]] = [f, gh - hg] + [g, hf - fh] + [h, fg - gf] \\ &= f(gh - hg) - (gh - hg) f + g(hf - fh) - (hf - fh) g + h(fg - gf) - (fg - gf) h \\ &= fgh - fhg - ghf + hgf + ghf - gfh - hfg + fhg + hfg - hgf - fgh + gfh = 0. \end{aligned}$$

Thus, (LA2) is satisfied, so we are done. □

Corollary 1.8. By Remark 1.3, $\mathfrak{gl}_n(F)$ with bracket given by $XY - YX$ is a Lie algebra.

1.3 Lie subalgebras

We'll now proceed by introducing some useful Lie algebra notions. The reader should be familiar with the fact that vector spaces have substructures known as vector subspaces, so it shouldn't be too surprisingly that Lie algebras have substructures known as *Lie subalgebras*.

Definition 1.9. [2, p.3] Let L be a Lie algebra with Lie bracket $[-,-]$. If $L' \subseteq L$ is a vector subspace satisfying $[x,y] \in L'$ for all $x, y \in L'$, then we call L' a *Lie subalgebra* of L.

Remark 1.10. Perhaps unsurprisingly, Lie subalgebras are themselves Lie algebras. Since $[x,y] \in L'$ for all $x, y \in L'$, we can define a Lie bracket $\ell : L'^2 \to L'$ on L' by $\ell(x,y) = [x,y]$. By doing so, L' satisfies (LA1) and (LA2), so it is a Lie algebra.

We'll revisit Lie subalgebras in just a moment, but before doing so we'll introduce the Lie algebra that is of central focus in this report: $\mathfrak{sl}_2(\mathbb{C})$.

Definition 1.11. [2, p.3] We define $\mathfrak{sl}_2(\mathbb{C})$ by

$$\mathfrak{sl}_2(\mathbb{C}) := \{x \in \mathfrak{gl}_2(\mathbb{C}) : \operatorname{tr}(x) = 0\},$$

where $\operatorname{tr}(x)$ denotes the trace of x.

Theorem 1.12. $\mathfrak{sl}_2(\mathbb{C})$ is a Lie algebra over $\mathbb{C}$.

Proof. Our strategy will be to show that $\mathfrak{sl}_2(\mathbb{C})$ is a Lie subalgebra of $\mathfrak{gl}_2(\mathbb{C})$. To begin, by Corollary 1.8, $\mathfrak{gl}_2(\mathbb{C})$ is a Lie algebra over $\mathbb{C}$. Hence, by Remark 1.10, it suffices to show that $\mathfrak{sl}_2(\mathbb{C})$ is a Lie subalgebra of $\mathfrak{gl}_2(\mathbb{C})$. To see that $\mathfrak{sl}_2(\mathbb{C})$ is a vector subspace of $\mathfrak{gl}_2(\mathbb{C})$, we refer to [5, Theorem 3.1.6] to get that

$\mathrm{tr}(\lambda x + y) = \mathrm{tr}(\lambda x) + \mathrm{tr}(y) = \lambda \mathrm{tr}(x) + \mathrm{tr}(y) = 0$ for each $\lambda \in \mathbb{C}$ and each $x, y \in \mathfrak{sl}_2(\mathbb{C})$. Next, by Definition 1.9, we must show that $[x, y] \in \mathfrak{sl}_2(\mathbb{C})$ for any $x, y \in \mathfrak{sl}_2(\mathbb{C})$, where $[x, y] := xy - yx$. To do so, we will first fix the $\mathfrak{sl}_2(\mathbb{C})$-basis $\{e, f, h\}$, as is done in [2, Exercise 1.12], where

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

It is sometimes convenient to use B_1, B_2, and B_3 in place of e, f, and h, so we will use these two notations synonymously for the remainder of this report. Now, for any $x, y \in \mathfrak{sl}_2(\mathbb{C})$, there exist $x_i, y_i \in \mathbb{C}$ for $1 \leq i \leq 3$ such that $x = \sum_{i=1}^{3} x_i B_i$ and $y = \sum_{i=1}^{3} y_i B_i$. Therefore, we have that

$$[x, y] = \left[\sum_{i=1}^{3} x_i B_i, \sum_{i=1}^{3} y_i B_i\right] \overset{\text{(BM1)}}{=} \sum_{i=1}^{3} x_i \left[B_i, \sum_{j=1}^{3} y_j B_j\right] \overset{\text{(BM2)}}{=} \sum_{i=1}^{3} x_i \sum_{j=1}^{3} y_j [B_i, B_j].$$

Hence, if $[B_i, B_j] \in \mathfrak{sl}_2(\mathbb{C})$ for all $1 \leq i, j \leq 3$, then $[x, y]$ is just a linear combination of matrices in $\mathfrak{sl}_2(\mathbb{C})$, so $[x, y] \in \mathfrak{sl}_2(\mathbb{C})$. Since $B_i, B_j \in \mathfrak{gl}_2(\mathbb{C})$ for all i, j, we immediately have that $[B_i, B_j] = 0$ whenever $i = j$. We'll proceed by computing $[e, f]$, $[e, h]$, and $[f, h]$.

$$\begin{aligned}
[e, f] = ef - fe &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = h; \\
[e, h] = eh - he &= \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & -1 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2 \\ 0 & 0 \end{pmatrix} = -2e; \\
[f, h] = fh - hf &= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix} = 2f.
\end{aligned}$$

Finally, we can apply Remark 1.5 to get that

$$\begin{aligned}
[f, e] &= -[e, f] = -h; \\
[h, e] &= -[e, h] = 2e; \\
[h, f] &= -[f, h] = -2f.
\end{aligned}$$

Thus, we have that $[B_i, B_j] \in \mathfrak{sl}_2(\mathbb{C})$ for all $1 \leq i, j \leq 3$. Hence, $[x, y] \in \mathfrak{sl}_2(\mathbb{C})$ for all $x, y \in \mathfrak{sl}_2(\mathbb{C})$, so $\mathfrak{sl}_2(\mathbb{C})$ is a Lie subalgebra of $\mathfrak{gl}_2(\mathbb{C})$. $\square$

1.4 Lie algebra homomorphisms

We'll conclude this first section of the report by discussing homomorphisms between Lie algebras. These homomorphisms are central to the notion of Lie algebra representations, which will be introduced in the following section.

Definition 1.13. [2, p.4] Let L_1 and L_2 be Lie algebras over a field F. Then, the map $\varphi : L_1 \to L_2$ is a *homomorphism* if φ is a linear map and $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for all $x, y \in L_1$. If, in addition to this, φ is bijective, then we call φ an *isomorphism*.

We'll now give a simple example of a Lie algebra homomorphism.

Example 1.14. If L_1 and L_2 are abelian Lie algebras over F and $\varphi : L_1 \to L_2$ is a linear map, then $\varphi\left([x, y]_{L_1}\right) = \varphi(0_{L_1}) = 0_{L_2}$ and $[\varphi(x), \varphi(y)]_{L_2} = 0_{L_2}$. Hence, φ is a Lie algebra homomorphism.

Before giving our next example, we'll show that it's sufficient to just check that φ preserves the Lie bracket on the basis vectors of L_1, which will make verifying whether φ is a Lie algebra homomorphism much easier.

Proposition 1.15. Let L_1 and L_2 be Lie algebras over a field F and let $\varphi : L_1 \to L_2$ be a linear map. Then, φ is a Lie algebra homomorphism if and only if

$$\varphi\left([b, b']\right) = \left[\varphi(b), \varphi(b')\right]$$

for each $b, b' \in \mathcal{B}$, where $\mathcal{B}$ is a basis for L_1.

Proof. ($\Longrightarrow$) This direction follows immediately from Definition 1.13.
($\Longleftarrow$) For the reverse direction, let $\mathcal{B} := \{A_k : k \in K\}$ be a basis for L_1, where K is some indexing set, and suppose that $\varphi([A_s, A_t]) = [\varphi(A_s), \varphi(A_t)]$ for any $s, t \in K$. Now, let $x, y \in L_1$. Then, we have indexing sets I, J such that $x = \sum_{i \in I} \lambda_i A_i$ and $y = \sum_{j \in J} \mu_j A_j$, where $\lambda_i, \mu_j \in F$ for each i, j. Thus, by the bilinearity of $[-, -]$, we have that

$$\varphi([x, y]) = \varphi\left(\left[\sum_{i \in I} \lambda_i A_i, \sum_{j \in J} \mu_j A_j\right]\right) = \varphi\left(\sum_{i \in I}\sum_{j \in J} \lambda_i \mu_j [A_i, A_j]\right).$$

Since φ is linear, this becomes

$$\sum_{i \in I}\sum_{j \in J} \lambda_i \mu_j \varphi([A_i, A_j]) = \sum_{i \in I}\sum_{j \in J} \lambda_i \mu_j [\varphi(A_i), \varphi(A_j)].$$

Then, by bilinearity of $[-, -]$ and linearity of φ, this becomes

$$\left[\sum_{i \in I} \lambda_i \varphi(A_i), \sum_{j \in J} \mu_j \varphi(A_j)\right] = \left[\varphi\left(\sum_{i \in I} \lambda_i A_i\right), \varphi\left(\sum_{j \in J} \mu_j A_j\right)\right] = [\varphi(x), \varphi(y)].$$

□

We'll now introduce a somewhat cryptic looking class of Lie algebra homomorphisms. Later, we'll uncover the fundamental relationship between this class of homomorphisms and the representations of $\mathfrak{sl}_2(\mathbb{C})$.

Example 1.16. [2, pp.67-68] We let $\mathbb{C}[X,Y]$ denote the vector space of polynomials in two variables X and Y with complex coefficients. Then, for each integer $d \geq 0$, we let V_d be the vector subspace of $\mathbb{C}[X,Y]$ consisting of all vectors of the form $\sum_{n=0}^{d} \lambda_n X^{d-n} Y^n$, where $\lambda_n \in \mathbb{C}$ for all $0 \leq n \leq d$. By this construction, it's easy to see that $\left\{X^{d-n}Y^n : 0 \leq n \leq d\right\}$ forms a basis for V_d. Now, for each d, we will define a Lie algebra homomorphism $\Psi_d : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V_d)$ by specifying how Ψ_d acts on the basis vectors of $\mathfrak{sl}_2(\mathbb{C})$ and extending linearly. Using the same $\mathfrak{sl}_2(\mathbb{C})$-basis as in Theorem 1.12, we define

$$\Psi_d(e) := X\frac{\partial}{\partial Y}; \quad \Psi_d(f) := Y\frac{\partial}{\partial X}; \quad \Psi_d(h) := X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}.$$

Theorem 1.17. [2, Theorem 8.1] $\Psi_d : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V_d)$ is a Lie algebra homomorphism.

Proof. [2, pp.68-69] By construction, Ψ_d is a linear map, so by Proposition 1.15, it suffices to show that $\Psi_d([B_i, B_j]) = [\Psi_d(B_i), \Psi_d(B_j)]$ for each $1 \leq i, j \leq 3$. To begin, if $i = j$, then

$$\Psi_d([B_i, B_j]) \overset{\text{(LA1)}}{=} \Psi_d\left(0_{\mathfrak{sl}_2(\mathbb{C})}\right) = 0_{\mathfrak{gl}(V_d)} \overset{\text{(LA1)}}{=} [\Psi_d(B_i), \Psi_d(B_j)].$$

Next, we want to verify the following:

$$\Psi_d([e,f]) = [\Psi_d(e), \Psi_d(f)]; \tag{1}$$
$$\Psi_d([e,h]) = [\Psi_d(e), \Psi_d(h)]; \tag{2}$$
$$\Psi_d([f,h]) = [\Psi_d(f), \Psi_d(h)]. \tag{3}$$

We mention that if $d = 0$, then (1), (2), and (3) trivially hold since $\Psi_0(\ell)$ is the zero map for every $\ell \in \mathfrak{sl}_2(\mathbb{C})$, so we'll assume that $d \geq 1$.

Starting with (1), we have that

$$\Psi_d([e,f]) = \Psi_d(h) = X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}.$$

Now,

$$\left(X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}\right) X^{d-n}Y^n = \begin{cases} dX^d, & \text{if } n = 0; \\ dY^d, & \text{if } n = d; \\ (d-2n)\, X^{d-n}Y^n, & \text{if } 1 < n < d. \end{cases}$$

Since $[\Psi_d(e), \Psi_d(f)] = \Psi_d(e) \circ \Psi_d(f) - \Psi_d(f) \circ \Psi_d(e)$, we have that

$$\begin{aligned}
[\Psi_d(e), \Psi_d(f)](X^d) &= X\frac{\partial}{\partial Y}\left(Y\frac{\partial}{\partial X}\left(X^d\right)\right) - Y\frac{\partial}{\partial X}\left(X\frac{\partial}{\partial Y}\left(X^d\right)\right) \\
&= X\frac{\partial}{\partial Y}\left(dX^{d-1}Y\right) - Y\frac{\partial}{\partial X}(0) \\
&= dX^d.
\end{aligned}$$

Similarly,

$$\begin{aligned}
[\Psi_d(e), \Psi_d(f)](Y^d) &= X\frac{\partial}{\partial Y}\left(Y\frac{\partial}{\partial X}\left(Y^d\right)\right) - Y\frac{\partial}{\partial X}\left(X\frac{\partial}{\partial Y}\left(Y^d\right)\right) \\
&= X\frac{\partial}{\partial Y}(0) - Y\frac{\partial}{\partial X}\left(dXY^{d-1}\right) \\
&= -dY^d.
\end{aligned}$$

Finally, we check the action on $X^{d-n}Y^n$, where $1 < n < d$, obtaining

$$\begin{aligned}
[\Psi_d(e), \Psi_d(f)]\left(X^{d-n}Y^n\right) &= X\frac{\partial}{\partial Y}\left(Y\frac{\partial}{\partial X}\left(X^{d-n}Y^n\right)\right) - Y\frac{\partial}{\partial X}\left(X\frac{\partial}{\partial Y}\left(X^{d-n}Y^n\right)\right) \\
&= X\frac{\partial}{\partial Y}\left((d-n)X^{d-n-1}Y^{n+1}\right) - Y\frac{\partial}{\partial X}\left(nX^{d-n+1}Y^{n-1}\right) \\
&= (d-n)(n+1)X^{d-n}Y^n - n(d-n+1)X^{d-n}Y^n \\
&= ((d-n)(n+1) - n(d-n+1))X^{d-n}Y^n \\
&= \left(dn + d - n^2 - n - nd + n^2 - n\right)X^{d-n}Y^n \\
&= (d-2n)X^{d-n}Y^n.
\end{aligned}$$

Hence, we've shown that $\Psi_d([e,f])$ and $[\Psi_d(e), \Psi_d(f)]$ agree on a basis of V_d, so they are the same linear map. Thus, (1) is true. In the proof of Theorem 1.12, we showed that $[f,e] = -[e,f]$. Therefore,

$$(1) \implies -\Psi_d([e,f]) = -[\Psi_d(e), \Psi_d(f)].$$

By the linearity of Ψ_d and Remark 1.5, we then get that

$$\Psi_d(-[e,f]) = \Psi_d([f,e]) = -[\Psi_d(e), \Psi_d(f)] = [\Psi_d(f), \Psi_d(e)].$$

For brevity, we will omit showing that (2) and (3) hold, though the approach is the same as how we proved (1). Moreover, in an identical fashion, we can use (2) and (3) respectively to show that $\Psi_d([h,e]) = [\Psi_d(h), \Psi_d(e)]$ and $\Psi_d([h,f]) = [\Psi_d(h), \Psi_d(f)]$. Hence, $\Psi_d([B_i, B_j]) = [\Psi_d(B_i), \Psi_d(B_j)]$ for each $1 \le i, j \le 3$, so we are done. □

2 Representations of Lie algebras

2.1 Representations

Having introduced enough Lie algebra machinery in the previous section, we have now arrived at a place in which we can talk about Lie algebra representations.

Definition 2.1. [2, pp.53-54] Let L be a Lie algebra over a field F. A *representation* of L is a Lie algebra homomorphism $\varphi : L \to \mathfrak{gl}(V)$, where V is a finite-dimensional vector space over F. Furthermore, if φ is injective, then we say that it is a *faithful* representation.

Example 2.2. $\Psi_d : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V_d)$ from Theorem 1.17 is a representation of $\mathfrak{sl}_2(\mathbb{C})$ for each $d \geq 0$.

Example 2.3. Adapted from [2, p.54], we have that $\iota : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}_2(\mathbb{C})$ given by $x \mapsto x$ is a faithful representation of $\mathfrak{sl}_2(\mathbb{C})$ since it's clearly injective and

$$\iota([x, y]) = xy - yx = [\iota(x), \iota(y)],$$

so it's a Lie algebra homomorphism.

2.2 Lie modules and submodules

In isolation, the following definition might seem entirely unrelated to the previous, so we'll provide some context first (motivated by [2, pp.55-56]). If we let $\varphi : L \to \mathfrak{gl}(V)$ be a representation of L, we have that $\varphi(\ell)$ is a linear map from $V \to V$ for each $\ell \in L$. Hence, $\varphi(\ell)(v) \in V$ for each $\ell \in L$ and each $v \in V$. This gives us a natural way of defining a map from $L \times V \to V$. However, this map has a nice property of interest. Since φ is a homomorphism, we have, by Definition 1.13, that

$$\varphi([x, y]) = [\varphi(x), \varphi(y)] = \varphi(x) \circ \varphi(y) - \varphi(y) \circ \varphi(x)$$

for all $x, y \in L$. By defining $\ell \cdot v := \varphi(\ell)(v)$ for each $\ell \in L$ and each $v \in V$, using the above equation, we have that

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v).$$

for all $x, y \in L$ and all $v \in V$. Through this lens, we have a clearer picture of how $\ell \in L$ "acts" on the vector space V. Appropriately, we refer to $\varphi(\ell)$ as the *action* of ℓ on V. This motivates our next definition.

Definition 2.4. [2, p.55] Let L be a Lie algebra over a field F. A *Lie module* for L, or L-module, is a finite-dimensional F-vector space V together with a bilinear map

$L \times V \to V : (\ell, v) \to \ell \cdot v$ satisfying

$$[x, y] \cdot v = x \cdot (y \cdot v) - y \cdot (x \cdot v) . \tag{LM1}$$

NB: When it's contextually obvious, we may omit the use of the $\cdot$ symbol. That is, we'll write ℓv instead of $\ell \cdot v$. Furthermore, we'll use the notation $\ell^k \cdot v$, where $l^0 \cdot v = v$ and $l^k \cdot v := \ell \cdot \left(\ell^{k-1} \cdot v\right)$ for $k \geq 1$.

Using what we've discussed above, one can take a representation $\varphi : L \to \mathfrak{gl}(V)$ of L and show that (LM1) is satisfied by defining $\ell \cdot v := \varphi(\ell)(v)$ for each $\ell \in L$ and each $v \in V$. It shouldn't come as much of a surprise that this map is bilinear (see [2, p.56]). This means that we can transform any representation of L into a Lie module for L. Naturally, one might ask whether we can do the reverse. That is, can we obtain a representation of L from a Lie module for L?

Proposition 2.5. [2, Exercise 7.2] Let L be a Lie algebra over a field F and V an L-module. Define $\varphi : L \to \mathfrak{gl}(V)$ by letting $\varphi(\ell)$ be the linear map $\varphi(\ell) : V \to V$ given by $v \mapsto \ell \cdot v$. Then, φ is a representation of L.

Proof. Firstly, $\varphi(\lambda\ell + k)$ is the linear map $(\lambda\ell + k) \cdot v$, where $\ell, k \in L$ and $\lambda \in F$. By bilinearity of $\cdot$, we have that

$$(\lambda\ell + k) \cdot v = \lambda(\ell \cdot v) + (k \cdot v) = \lambda\varphi(\ell) + \varphi(k),$$

so φ is a linear map. Next, consider $\varphi([x, y])$ for $x, y \in L$. By construction, $\varphi([x, y])$ is the linear map $v \mapsto [x, y] \cdot v$. Hence, we have that

$$[x, y] \cdot v \overset{\text{(LM1)}}{=} x \cdot (y \cdot v) - y \cdot (x \cdot v) = \varphi(x) \circ \varphi(y) - \varphi(y) \circ \varphi(x) = [\varphi(x), \varphi(y)],$$

so φ is a Lie algebra homomorphism. □

This correspondence between Lie modules and representations means that we can use the two notions interchangeably.

2.3 Submodules and irreducible modules

We'll now introduce some Lie modules constructs.

Definition 2.6. [2, p.57] Let V be a Lie module for the Lie algebra L. A *submodule* of V is a subspace W of V such that $\ell \cdot w \in W$ for each $\ell \in L$ and each $w \in W$.

Given any Lie module V for the Lie algebra L, we clearly have that $\{0_V\}$ and V are subspaces of V. Trivially, V is a submodule of itself, and $\ell \cdot 0_V = 0_V \in \{0_V\}$ for any

$\ell \in L$, so $\{0_V\}$ is a submodule of V. Hence, we'll refer to these submodules as the *trivial submodules* of V. This leads us to our next definition.

Definition 2.7. [2, p.58] The Lie module V is said to be *irreducible* if V is non-zero and its only submodules are trivial.

We'll return to irreducible modules shortly, but firstly we'll introduce another definition.

Definition 2.8. [2, Exercise 7.3] Let V be a module for the Lie algebra L. For any $v \in V$, we define the *submodule generated by* v, which we denote S_v, to be the span of all elements of the form

$$\ell_1 \cdot (\ell_2 \cdot \ldots \cdot (\ell_n \cdot v) \ldots),$$

where $\ell_1, \ldots, \ell_n \in L$ and $n \geq 1$.

To illustrate this definition, we'll provide a short example.

Example 2.9. Consider the module V_2 for $\mathfrak{sl}_2(\mathbb{C})$. We'll proceed to determine to submodule generated by $X^2 \in V_2$. Firstly, we'll fix the standard basis $\{e, f, h\}$ for $\mathfrak{sl}_2(\mathbb{C})$. We then have that $f \cdot X^2 = 2XY$, so $f \cdot \left(f \cdot X^2\right) = 2Y^2$. Furthermore, we have that $h \cdot X^2 = 2X^2$. Since $\left\{X^2, XY, Y^2\right\}$ forms a basis for V_2, it follows that

$$\operatorname{span}\left\{2X^2, 2XY, 2Y^2\right\} = \operatorname{span}\left\{X^2, XY, Y^2\right\} = V_2.$$

Thus, the submodule generated by X^2 is V_2 itself.

While this might seem like a lucky coincidence, the following lemma will prove otherwise; which was motivated by generalising [2, Exercise 8.1].

Lemma 2.10. Consider the module V_d for $\mathfrak{sl}_2(\mathbb{C})$ with $d \geq 1$. Then, for any non-zero $v \in V_d$, the submodule generated by v is V_d.

Proof. Let $\{e, f, h\}$ denote the standard basis for $\mathfrak{sl}_2(\mathbb{C})$ and fix the basis $\mathcal{B} := \left\{X^d, X^{d-1}Y, \ldots, XY^{d-1}, Y^d\right\}$ for V_d. Now, by letting $v \neq 0_{V_d} \in V_d$, we have that v can be expressed as a linear combination of vectors in $\mathcal{B}$. Let X^aY^b, where $a + b = d$, be the basis vector in this linear combination with non-zero coefficient, say λ, such that a is maximal. Now, if $a \neq 0$, then

$$e^j \cdot (f^a \cdot v) = \frac{a!d!}{(d-j)!} \lambda X^j Y^{d-j}$$

for each $0 \leq j \leq d$. That is, for each j, we generate a unique non-zero scalar multiple of an element of $\mathcal{B}$, so $\operatorname{span}\left\{e^j \cdot (f^a \cdot v) : 0 \leq j \leq d\right\} = V_d$. On the other hand, if $a = 0$, then we must have that $v = \lambda Y^d$, in which case

$$e^j \cdot v = \frac{d!}{(d-j)!} \lambda X^j Y^{d-j}$$

for $0 \leq j \leq d$. Hence, by the same argument, $\operatorname{span}\left\{e^j \cdot v : 0 \leq j \leq d\right\} = V_d$. $\square$

We'll now show that there is a connection between irreducible modules and generated submodules.

Lemma 2.11. (Modified version of [2, Exercise 7.3]) Let V be a module for the Lie algebra L. If for any non-zero $v \in V$ we have that $S_v = V$, then V is irreducible.

Proof. Suppose that $S_v = V$ for every non-zero $v \in V$ and that V is *not* irreducible. Then, there exists a non-trivial submodule W of V. By Definition 2.6, we have that $\ell \cdot w \in W$ for every $\ell \in L$ and every $w \in W$. Therefore, $\ell_1 \cdot (\ell_2 \cdot \ldots \cdot (\ell_n \cdot w) \ldots) \in W$ for every $\ell_1, \ldots, \ell_n \in L$ and $w \in W$. Hence, if we take w to be non-zero, S_w must be a subspace of W, but $S_w = V$, giving contradiction. Therefore, such a W doesn't exist, so V is irreducible. $\square$

By combining the previous two lemmas, we can now immediately prove the following theorem.

Theorem 2.12. [2, Theorem 8.2] The $\mathfrak{sl}_2(\mathbb{C})$-module V_d is irreducible for every $d \geq 0$

Proof. By Lemma 2.10, if $d \geq 1$, then $S_v = V_d$ for any non-zero $v \in V_d$, so V_d is irreducible by Lemma 2.11. Then, since $\ell \cdot v = 0$ for every $\ell \in \mathfrak{sl}_2(\mathbb{C})$ and $v \in V_0$, we have that the only submodules of V_0 are V_0 itself and $\{0\}$, so V_0 is irreducible. $\square$

We'll now conclude this section with a definition which alludes to our next objective.

Definition 2.13. [4, p.25] Let V and W be Lie modules for the Lie algebra L. Then, a *Lie module homomorphism* from V to W is a linear map $\theta : V \to W$ such that $\theta(\ell \cdot v) = \ell \cdot \theta(v)$ for all $v \in V$ and all $\ell \in L$. If θ is bijective, then it is an isomorphism of Lie modules.

3 Classifying $\mathfrak{sl}_2(\mathbb{C})$-modules

3.1 Irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules

This is a good place to pause briefly to discuss what we've been working towards so far. We now know that $\mathfrak{sl}_2(\mathbb{C})$ has an infinite family of irreducible modules: the V_d. The ultimate goal would be to classify (i.e. up to isomorphism) *every* $\mathfrak{sl}_2(\mathbb{C})$-module. A slightly less lofty goal is to classify just the irreducible modules, though this will require some work. We'll start things off with a simple result.

Proposition 3.1. [2, p.71] Suppose that $d_1 \neq d_2$. Then, we have that $V_{d_1} \ncong V_{d_2}$.

Proof. [2, p.71] Since $\dim(V_{d_1}) = d_1 + 1 \neq \dim(V_{d_2}) = d_2 + 1$, V_{d_1} and V_{d_2} are not isomorphic as vector spaces. Clearly any Lie module isomorphism is a vector space isomorphism, so we must have that $V_{d_1} \ncong V_{d_2}$. □

Thus, each V_d lies in a distinct isomorphism class. What is not at all obvious is that *every* irreducible $\mathfrak{sl}_2(\mathbb{C})$-module is isomorphic to one of the V_d. To prove this, we'll first need some results relating to the eigenvectors of the action of $h \in \mathfrak{sl}_2(\mathbb{C})$.

To provide a little context, if we have a representation φ of $\mathfrak{sl}_2(\mathbb{C})$, then $\varphi(h)$ is a linear map from V to V. Therefore, we can represent $\varphi(h)$ as an $n \times n$ matrix $H \in \mathfrak{gl}_n(\mathbb{C})$, where $n = \dim(V)$. By the Fundamental Theorem of Algebra, H has at least one eigenvalue in $\mathbb{C}$, hence at least one eigenvector. To proceed, the first thing that we're going to do is investigate the actions of e and f on an arbitrary H-eigenvector.

Lemma 3.2. [2, Lemma 8.3] Let V be an $\mathfrak{sl}_2(\mathbb{C})$-module and suppose that $v \in V$ is an eigenvector of H with eigenvalue $\lambda \in \mathbb{C}$. Then, we have that:

(1) Either $e \cdot v = 0_V$ or $e \cdot v$ is an eigenvector of H with eigenvalue $\lambda + 2$.

(2) Either $f \cdot v = 0_V$ or $f \cdot v$ is an eigenvector of H with eigenvalue $\lambda - 2$.

Proof. [2, p.71] For **(1)**, by (LM1), we have that $h \cdot (e \cdot v) = e \cdot (h \cdot v) + [h, e] \cdot v$. Now, since v is an eigenvector of H with eigenvalue λ, we have that $h \cdot v = \lambda v$. Then, since $[h, e] = 2e$, we obtain

$$h \cdot (e \cdot v) = e \cdot (\lambda v) + 2e \cdot v \overset{\text{(BM2)}}{=} \lambda e \cdot v + 2e \cdot v \overset{\text{(BM1)}}{=} (\lambda + 2)\, e \cdot v,$$

which implies **(1)**. For **(2)**, the calculation is very similar. By (LM1), we have that $h \cdot (f \cdot v) = f \cdot (h \cdot v) + [h, f] \cdot v$. Since $[h, f] = -2f$, we get that

$$h \cdot (f \cdot v) = e \cdot (\lambda v) - 2f \cdot v \overset{\text{(BM2)}}{=} \lambda f \cdot v - 2f \cdot v \overset{\text{(BM1)}}{=} (\lambda - 2)\, f \cdot v,$$

which implies **(2)**. □

This result shows that it's possible for H to have an eigenvector v such that $e \cdot v = 0_V$ or $f \cdot v = 0_V$. Next, we'll show that H *always* has eigenvectors w_1 and w_2 such that $e \cdot w_1 = 0_V$ and $f \cdot w_2 = 0_V$.

Lemma 3.3. (Extension of [2, Lemma 8.4]) Let V be an $\mathfrak{sl}_2(\mathbb{C})$-module. Then, V contains H-eigenvectors w_1 and w_2 such that $e \cdot w_1 = 0_V$ and $f \cdot w_2 = 0_V$.

Proof. [2, p.71] Suppose that $v \in V$ is a H-eigenvector with eigenvalue λ. Then, consider the following two infinite sequences of vectors:

$$v,\ e \cdot v,\ e^2 \cdot v,\ e^3 \cdot v,\ \ldots \tag{†}$$

$$v,\ f \cdot v,\ f^2 \cdot v,\ f^3 \cdot v,\ \ldots \tag{‡}$$

If each vector in (†) is non-zero, then, by Lemma 3.2, each vector in (†) is a H-eigenvector with eigenvalue $\lambda,\ \lambda+2,\ \lambda+4,\ \lambda+6,\ \ldots$ respectively. Since each of these eigenvalues is distinct, this implies that (†) is an infinite sequence of linearly independent H-eigenvectors. However, V is finite-dimensional, so this is absurd. Thus, there must exist some $k \geq 0$ such that $e^k \cdot v \neq 0_V$ and $e^{k+1} \cdot v = 0_V$. By setting $w_1 := e^k \cdot v$, we get that $h \cdot w_1 = (\lambda + 2k)\, w_1$ and $e \cdot w_1 = 0_V$, as required.

Similarly, if each vector in (‡) is non-zero, then we have an infinite sequence of linearly independent H-eigenvectors with eigenvalues $\lambda,\ \lambda-2,\ \lambda-4,\ \lambda-6,\ \ldots$ respectively. Therefore, there exists some $j \geq 0$ such that $f^j \cdot v \neq 0_V$ and $f^{j+1} \cdot v = 0_V$. Upon setting $w_2 := f^j \cdot v$, we have that $h \cdot w_2 = (\lambda - 2j)\, w_2$ and $f \cdot w_2 = 0_V$. ☐

This result shows that, for a given $\mathfrak{sl}_2(\mathbb{C})$-module V, the set $\left\{f^i \cdot v : 0 \leq i \leq d\right\}$ is a set of non-zero H-eigenvectors, where $d+1$ is the smallest positive integer such that $f^{k+1} \cdot v = 0_V$. We will now show that this set of vectors forms a basis for V.

Lemma 3.4. [2, p.72] *Let V be an irreducible $\mathfrak{sl}_2(\mathbb{C})$-module, $v \in V$ a H-eigenvector of eigenvalue λ, and $d+1$ the smallest positive integer such that $f^{d+1} \cdot v = 0_V$. Then, $\mathcal{B} := \left\{f^i \cdot v : 0 \leq i \leq d\right\}$ forms a basis for V and $\lambda = d$.*

Proof. [2, p.72] By Lemma 3.2 **(2)**, we know that each of the $b \in \mathcal{B}$ are H-eigenvectors with distinct eigenvalues, so they are linearly independent over V. Therefore, $\operatorname{span}(\mathcal{B})$ is a subspace of V. By Definition 2.6, for $\operatorname{span}(\mathcal{B})$ to be a submodule of V, we require that $\ell \cdot b \in \operatorname{span}(\mathcal{B})$ for each $\ell \in \mathfrak{sl}_2(\mathbb{C})$ and $b \in \operatorname{span}(\mathcal{B})$. By the bilinearity of $\cdot$, it suffices to check that $\ell \cdot b \in \operatorname{span}(\mathcal{B})$ for each $\ell \in \{e, f, h\}$ and $b \in \mathcal{B}$. Since each $b \in \mathcal{B}$ is a H-eigenvector, we clearly have that $h \cdot b \in \operatorname{span}(\mathcal{B})$ for each $b \in \mathcal{B}$. Furthermore, $f \cdot b \in \operatorname{span}(\mathcal{B})$ for each $b \in \mathcal{B}$ by construction. Thus, it remains to show that $e \cdot b \in \operatorname{span}(\mathcal{B})$ for each $b \in \mathcal{B}$. To do so, we will prove, by induction on i, that

$$e \cdot \left(f^i \cdot v\right) \in \operatorname{span}(\mathcal{B}),$$

where $v \in V$ is a H-eigenvector with eigenvalue $\lambda \neq 0$. Firstly, if $i = 0$, then $e \cdot \left(f^i \cdot v\right) = e \cdot v = 0_V \in \operatorname{span}(\mathcal{B})$. For the inductive step, suppose that $e \cdot \left(f^{i-1} \cdot v\right) \in \operatorname{span}(\mathcal{B})$ for some $1 \leq i-1 < d$. By considering $e \cdot \left(f^i \cdot v\right)$, we have that

$$e \cdot \left(f^i \cdot v\right) = e \cdot \left(f \cdot \left(f^{i-1} \cdot v\right)\right).$$

Letting $u := f^{i-1} \cdot v$, this becomes

$$e \cdot (f \cdot u) \overset{\text{(LM1)}}{=} f \cdot (e \cdot u) + [e, f] \cdot u = f \cdot (e \cdot u) + h \cdot u.$$

By the inductive hypothesis, $e \cdot u \in \operatorname{span}(\mathcal{B})$, so $f \cdot (e \cdot u) \in \operatorname{span}(\mathcal{B})$ since $e \cdot u$ can be written as a linear combination of vectors in $\mathcal{B}$ and $f \cdot b \in \operatorname{span}(\mathcal{B})$ for each $b \in \mathcal{B}$. Furthermore, by the proof of Lemma 3.3, $h \cdot u = (\lambda - 2(i-1))\, u \in \operatorname{span}(\mathcal{B})$, thus

$$f \cdot (e \cdot u) + h \cdot u = e \cdot \left(f^i \cdot v\right) \in \operatorname{span}(\mathcal{B}).$$

Therefore, $e \cdot b \in \operatorname{span}(\mathcal{B})$ for each $b \in \mathcal{B}$, so $\mathcal{B}$ forms a basis for a submodule of V. Then, since V is irreducible and $v \neq 0_V$, this submodule must be V itself.

Now, to show that $\lambda = d$, we first observe that H, with respect to the V-basis $\mathcal{B}$, is given by the $d+1 \times d+1$ matrix

$$\begin{pmatrix} \lambda & 0 & \dots & 0 & 0 \\ 0 & \lambda - 2 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \lambda - 2(d-1) & 0 \\ 0 & 0 & \cdots & 0 & \lambda - 2d \end{pmatrix} \in \mathfrak{gl}_{d+1}(\mathbb{C}).$$

Hence, $\operatorname{tr}(H) = \sum_{i=0}^{d} \lambda - 2i = (d+1)\lambda - (d+1)\, d$. Now, we can introduce matrices $E, F \in \mathfrak{gl}_{d+1}(\mathbb{C})$ corresponding to the actions of e and f on $\mathcal{B}$. Since $H = [E, F]$, by using the properties of the trace in [5, Theorem 3.1.6], we have that

$$\operatorname{tr}(H) = \operatorname{tr}(EF - FE) = \operatorname{tr}(EF) - \operatorname{tr}(EF) = 0.$$

Thus, $\operatorname{tr}(H) = (d+1)\lambda - (d+1)\, d = 0$, so we must have that $\lambda = d$. □

It's easy to see that $|\mathcal{B}| = |V_d| = d+1$. Therefore, we have that V and V_d are isomorphic as vector spaces. To show that V and V_d are isomorphic as Lie modules, we will find an explicit isomorphism.

Theorem 3.5. [2, p.72] If V is an irreducible $\mathfrak{sl}_2(\mathbb{C})$-module, then $V \cong V_d$ for some $d \geq 0$.

Proof. [2, pp.71-72] Applying Lemma 3.4, we get that V has basis $\left\{f^k \cdot v : 0 \leq k \leq d\right\}$. Furthermore, by repeatedly applying f to X^d, we can see that

$$f^k \cdot X^d = \frac{d!}{(d-k)!} X^{d-k} Y^k$$

for each $0 \leq k \leq d$. Since $f^k \cdot X^d$ is always a scalar multiple of a unique vector in the standard V_d-basis $\left\{X^n Y^{d-n} : 0 \leq n \leq d\right\}$, we have that $\{f^k \cdot X^d : 0 \leq k \leq d\}$ is a basis for V_d. In Lemma 3.3, we saw that $f^k \cdot v$ is a H-eigenvector with eigenvalue $\lambda - 2k$ for each k. Furthermore, by Lemma 3.4, we have that $k = d$, so $f^k \cdot v$ has eigenvalue $d - 2k$ for each k. Next, we proceed by computing $h \cdot \left(f^k \cdot X^d\right)$.

$$h \cdot \left(f^k \cdot X^d\right) = h \cdot \left(\frac{d!}{(d-k)!} X^{d-k} Y^k\right) = (d-2k) \frac{d!}{(d-k)!} X^{d-k} Y^k = (d-2k)\, f^k \cdot X^d.$$

Thus, $f^k \cdot X^d$ and $f^k \cdot v$ are both H-eigenvectors with the same eigenvalue. Now, we want to define a Lie module isomorphism $\theta : V \to V_d$. By Definition 2.13, θ must commute with the action of h. For this to be the case, θ must map H-eigenvectors to H-eigenvectors with the same eigenvalue. Hence, we define θ by $\theta\left(f^k \cdot v\right) = f^k \cdot X^d$ for each $0 \leq k \leq d$ and extending linearly. Since V and V_d have the same dimension, θ is clearly a vector space isomorphism that, by construction, commutes with the actions of f and h. For θ to be a Lie module isomorphism, it must also commute with the action of e. Therefore, we will show that

$$e \cdot \theta\left(f^k \cdot v\right) = \theta\left(e \cdot \left(f^k \cdot v\right)\right) \tag{$*$}$$

for each $0 \leq k \leq d$ by induction on k. For the base case, let $k = 0$. Then,

$$e \cdot \theta\left(f^k \cdot v\right) = e \cdot \theta(v) = e \cdot X^d = 0_{V_d}$$

and

$$\theta\left(e \cdot \left(f^k \cdot v\right)\right) = \theta(e \cdot v) = \theta(0_V) = 0_{V_d}.$$

Hence, $(*)$ is satisfied. For the inductive step, suppose that $(*)$ holds for some $1 \leq k < d$. Then, we have that

$$\begin{aligned} \theta\left(e \cdot \left(f^{k+1} \cdot v\right)\right) = \theta\left(e \cdot \left(f \cdot \left(f^k \cdot v\right)\right)\right) &\overset{\text{(LM1)}}{=} \theta\left(f \cdot \left(e \cdot \left(f^k \cdot v\right)\right) + [e,f] \cdot \left(f^k \cdot v\right)\right) \\ &= f \cdot \theta\left(e \cdot \left(f^k \cdot v\right)\right) + h \cdot \theta\left(f^k \cdot v\right). \end{aligned} \tag{$\dagger$}$$

By the inductive hypothesis,

$$\begin{aligned} f \cdot \theta\left(e \cdot \left(f^k \cdot v\right)\right) = f \cdot \left(e \cdot \left(\theta\left(f^k \cdot v\right)\right)\right) &\overset{\text{(LM1)}}{=} e \cdot \left(f \cdot \left(\theta\left(f^k \cdot v\right)\right)\right) + [f,e] \cdot \theta\left(f^k \cdot v\right) \\ &= e \cdot \theta\left(f^{k+1} \cdot v\right) - h \cdot \theta\left(f^k \cdot v\right). \end{aligned} \tag{$\ddagger$}$$

Combining $(\dagger)$ and $(\ddagger)$, we obtain

$$\theta\left(e \cdot \left(f^{k+1} \cdot v\right)\right) = e \cdot \theta\left(f^{k+1} \cdot v\right).$$

Thus, by induction, $(*)$ holds for all $0 \leq k \leq d$. Now, since θ commutes with the actions of e, f, and h, by the bilinearity of $\cdot$, $\ell \cdot \theta(v) = \theta(\ell \cdot v)$ for every $\ell \in L$ and $v \in V$, so θ defines an isomorphism between V and V_d. □

3.2 Completely reducible $\mathfrak{sl}_2(\mathbb{C})$-modules

We have successfully classified all of the finite-dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$-modules. To bring things to a close, we will introduce a new class of $\mathfrak{sl}_2(\mathbb{C})$-modules and briefly talk about their classification.

Proposition 3.6. (Motivated by [6, p.10]) Let U and W be L-modules. Then, $U \oplus W$ — with $\oplus$ referring to the *direct sum* of U and V as vector spaces — is an L-module, where we define

$$\ell \cdot (u, w) = (\ell \cdot u,\, \ell \cdot w)$$

for each $\ell \in L$, each $u \in U$, and each $v \in V$.

Proof. We must verify that (LM1) is satisfied. Hence, by letting $(u, w) \in U \oplus W$ and $x, y \in \mathfrak{sl}_2(\mathbb{C})$, we get that

$$[x, y] \cdot (u, w) = (xy - yx) \cdot (u, w) = ((xy - yx) \cdot u,\, (xy - yx) \cdot w)\,. \qquad (\dagger)$$

Since U and V are $\mathfrak{sl}_2(\mathbb{C})$-modules, we can apply (LM1) to both components to obtain

$$\begin{aligned}(\dagger) &= ((xy)\,u - (yx)\,u,\, (xy)\,v - (yx)\,v) = ((xy)\,u,\, (xy)\,v) - ((yx)\,u + yx\,(v)) \\ &= (xy)\,(u, w) - (yx)\,(u, w) = (xy - yx) \cdot (u, w)\,.\end{aligned}$$

Thus, (LM1) is satisfied, so $U \oplus W$ is an L-module. $\square$

This notion motivates our next definition.

Definition 3.7. [2, p.59] Let V be an L-module. Then, we say that V is *completely reducible* if $V \cong \bigoplus_{i=1}^{n} U_i$, where each U_i is an irreducible L-module and $n \geq 1$.

In the case of $\mathfrak{sl}_2(\mathbb{C})$, as we proved in Theorem 3.5, every irreducible $\mathfrak{sl}_2(\mathbb{C})$-module is isomorphic to some V_d. Hence, if V is a completely reducible $\mathfrak{sl}_2(\mathbb{C})$-module, then V can be written as a direct sum of the V_d modules. What is perhaps a surprising result is that *every* finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module is completely reducible. This is a special case of Weyl's Theorem [2, Theorem 8.7], which we will not be proving.

Theorem 3.8. If V is a finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module, then V can be written as a direct sum of finitely many V_d modules.

This result truly showcases the importance of the V_d modules — they fully control the structure of all of the finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-modules.

⋆

Closing Remarks

Report Summary

We'll conclude this report by summarising what we've covered, then we'll discuss a few possible topics for further study.

(1) $\mathfrak{sl}_2(\mathbb{C})$ is the Lie algebra consisting of 2×2 complex matrices with zero trace, with Lie bracket given by $[x, y] := xy - yx$. The set $\{e, f, h\}$ forms a basis for $\mathfrak{sl}_2(\mathbb{C})$, where

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}; \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}; \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(2) $\mathfrak{gl}(V)$ is the Lie algebra consisting of all linear maps from V to V, where V is a finite-dimensional F-vector space, with Lie bracket given by $[f, g] := f \circ g - g \circ f$. Furthermore, $\mathfrak{gl}(V)$ is isomorphic to $\mathfrak{gl}_n(F)$ — the vector space of $n \times n$ matrices with entries in F, where $n = \dim(V)$.

(3) A Lie algebra *homomorphism* between Lie algebras L_1 and L_2 is a linear map $\varphi : L_1 \to L_2$ such that $\varphi([x, y]) = [\varphi(x), \varphi(y)]$ for each $x, y \in L_1$.

(4) The map $\Psi_d : \mathfrak{sl}_2(\mathbb{C}) \to \mathfrak{gl}(V_d)$ is a Lie algebra homomorphism for each $d \geq 0$. We define Ψ_d by

$$\Psi_d(e) := X\frac{\partial}{\partial Y}; \quad \Psi_d(f) := Y\frac{\partial}{\partial X}; \quad \Psi_d(h) := X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}.$$

(5) A *representation* of the Lie algebra L is a Lie algebra homomorphism of the form $\varphi : L \to \mathfrak{gl}(V)$, where V is a finite-dimensional vector space. Our canonical example is Ψ_d, which is a representation of $\mathfrak{sl}_2(\mathbb{C})$ for each $d \geq 0$.

(6) Representations can be transformed into Lie modules and vice-versa.

(7) *Irreducible* modules are non-zero modules whose only submodules are trivial.

(8) Every finite-dimensional irreducible $\mathfrak{sl}_2(\mathbb{C})$-module is isomorphic to one of the V_d modules.

(9) An L-module is *completely reducible* if it is isomorphic to a direct sum of finitely many irreducible L-modules.

(10) Weyl's Theorem states that every finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-module is completely reducible, hence can be written as a direct sum of the V_d modules.

Further Directions

* A very natural extension of this report would be to prove Weyl's Theorem for $\mathfrak{sl}_2(\mathbb{C})$. We could then look into how one might prove the more general case of Weyl's Theorem which applies to any complex semisimple Lie algebra.

* As we stated in our introduction, every Lie group has an associated Lie algebra. The Lie group associated to $\mathfrak{sl}_2(\mathbb{C})$ is $\mathrm{SL}_2(\mathbb{C})$ — the group of 2×2 complex matrices with determinant 1, with the group operation given by matrix multiplication. We could hence investigate the connections between the representations of $\mathfrak{sl}_2(\mathbb{C})$ and the representations of its underlying Lie group.

* When we defined a Lie module back in Definition 2.4, we stated that V was a finite-dimensional vector space. We could extend this definition by permitting V to be infinite-dimensional and examine some examples of infinite-dimensional $\mathfrak{sl}_2(\mathbb{C})$-modules. Interestingly, not only does $\mathfrak{sl}_2(\mathbb{C})$ have infinite-dimensional modules, but it has infinite-dimensional irreducible modules. An example of this, belonging to class of modules known as *Verma modules*, is given in [2, Example 15.12].

Acknowledgements

I'd like to extend my thanks to my supervisor, Vanessa, for her helpful feedback and for proposing the topic of this report.

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