Representation theory of $\mathfrak{sl}_2(\mathbb{C})$

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- $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for any $x, y, z \in L$.

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Then, L, together with the Lie bracket, is a **Lie algebra** over F.

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Both of these Lie algebras have their Lie bracket defined by

$$[x, y] := xy - yx.$$

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$\mathfrak{sl}_2(\mathbb{C})$ – 2×2 matrices with zero trace with entries in $\mathbb{C}$.

Remark: We can form a basis of $\mathfrak{sl}_2(\mathbb{C})$ using the following matrices:

$$e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix},\ f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},\ h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

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Then, M, together with this map, is called a **Lie module** for L.

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Definition: Irreducible Lie module

If M is a non-zero Lie module and has no submodules other than $\{0\}$ and M, then M is said to be **irreducible**.

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Example 2: V_d has the following basis vectors:

$$X^d, X^{d-1}Y, \ldots, XY^{d-1}, Y^d.$$

V_d as an $\mathfrak{sl}_2(\mathbb{C})$ module

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How does $\mathfrak{sl}_2(\mathbb{C})$ "act" on V_d?

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How does $\mathfrak{sl}_2(\mathbb{C})$ "act" on V_d?

Remark: It suffices to consider how the basis vectors of $\mathfrak{sl}_2(\mathbb{C})$ act on the basis vectors of V_d.

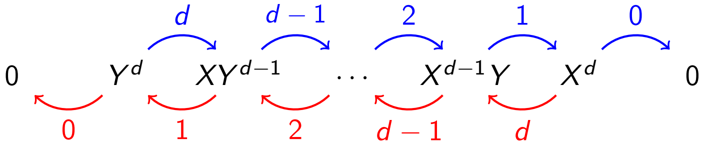
The action on V_d

$$0 \qquad Y^d \qquad XY^{d-1} \qquad \cdots \qquad X^{d-1}Y \qquad X^d \qquad 0$$

The action on V_d

$$0 \qquad Y^d \xrightarrow{d} XY^{d-1} \xrightarrow{d-1} \cdots \xrightarrow{2} X^{d-1}Y \xrightarrow{1} X^d \xrightarrow{0} 0$$

The action on V_d



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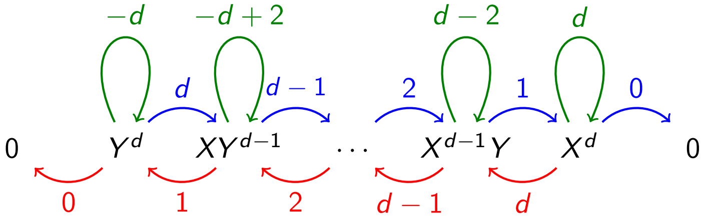

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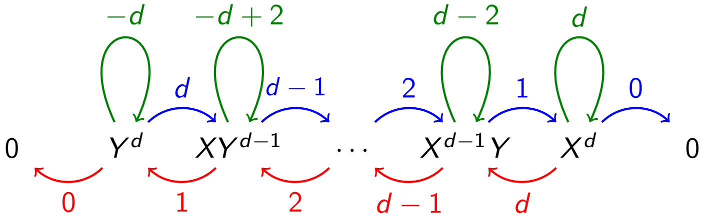

$$e \to X\frac{\partial}{\partial Y}; \quad f \to Y\frac{\partial}{\partial X}; \quad h \to X\frac{\partial}{\partial X} - Y\frac{\partial}{\partial Y}.$$

Classifying finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$ modules

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Theorem

- V_d is an irreducible $\mathfrak{sl}_2(\mathbb{C})$ module.
- If M is a finite-dimensional $\mathfrak{sl}_2(\mathbb{C})$ module, then M is isomorphic to one of the V_d.

Thank you for your attention.

Please feel free to ask any questions.